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INTEGRABILITY IN FINITE TERMS AND ACTIONS OF LIE GROUPS

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Dedicated to Yulii Iliashenko on his 75th birthday

ABSTRACT. According to Liouville's Theorem, an idefinite integral of an elementary function is usually not an elementary function. In these notes, we discuss that statement and a proof of this result. The differential Galois group of the extension obtained by adjoining an integral does not determine whether the integral is an elementary function or not. Nevertheless, Liouville's Theorem can be proved using differential Galois groups. The first step towards such a proof was suggested by Abel. This step is related to algebraic extensions and their finite Galois groups. A significant part of these notes is dedicated to the second step dealing with pure transcendent extensions and their Galois groups, which are connected Lie groups. The idea of the proof goes back to J. Liouville and J. F. Ritt.

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INTRODUCTION

Let K be a subfield of the field of meromorphic functions on a connected domain U of the complex line closed under the differentiation (i.e., if $f \in K$ then $f' \in K$). Such field K with the operation of differentiation $f \to f'$ provides an example of *functional differential field*. Liouville's Theorem suggests conditions on a function f from a function differential field K which are necessary and sufficient for representability of an indefinite integral of f in *generalized elementary functions over* K.

In Sections 1–4 we define the notions of functional differential field K and generalized elementary functions over K (we follow here the exposition from [1]). A natural definition of generalized elementary functions over K (see Definitions 4 and 5 below) is hard to deal with. In particular it involves a big enough list of *basic elementary functions* and makes use of a non-algebraic operation of composition of

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functions. An algebraic definition (see Definition 1 below) of *generalized elementary extension* of K uses solution of the simplest differential equations instead of composition of functions. We explain how this natural definition can be reduced to the algebraic one. In Section 5 we state Liouville's Theorem and outline its inductive proof.

Using the algebraic definition of generalized elementary extension one can generalize Liouville's Theorem for an abstract differential field K, whose elements are not necessarily meromorphic functions. An exposition of this result and references to original papers can be found in [1]. This abstract algebraic result is not directly applicable to integrals of elementary functions of one complex variable, which could be multivalued, could have singularities, and so on. For its applications to elementary functions extra arguments (analogous to arguments we presented in Sections 1–4) are needed.

The differential Galois group of the extension $K \subset K(y)$ does not contain enough information to determine if the integral y belongs to a generalized elementary extension of K or not. Indeed, if the integral y does not belong to K then the differential Galois group of $K(y)$ over K is always the same: it is isomorphic to the additive group of complex numbers. From this fact one can conclude that the Galois theory is not sensitive enough for proving Liouville's Theorem.

The goal of this notes is to show that Galois theory type arguments allow to prove Liouville's Theorem.

The first step towards such proof was suggested by Abel. This step is related to finite algebraic extensions and their finite Galois groups (see Section 6).

The second step deals with a pure transcendental extension F of a functional differential field K, obtained by adjoining $k+n$ logarithms and exponentials, algebraically independent over K (see Section 7). The differential Galois group of the extension $K \subset F$ is a $(k+n)$-dimensional connected commutative algebraic group G. It is naturally represented as a group of analytic automorphisms of an analytic variety X. Thus G acts not only on the differential field F but also on other objects such as closed 1-forms on X. This action plays a key role in the proof.

The idea of the proof goes back to Liouville. I came up with it trying to understand and to comment the classical book written by J. Ritt [2]. I am grateful to Michael Singer who invited me to write comments for a new edition of this book.

1. Abstract Differential Fields

A field F is said to be a *differential field* if an additive map $a \to a'$ is fixed that satisfies the Leibnitz rule $(ab)' = a'b + ab'$. The element a' is called the *derivative* of a. An element $y \in F$ is called a *constant* if $y' = 0$.

All constants in F form the *field of constants*. We add to the definition of differential field an extra condition that *the field of constants is the field of complex numbers*(for our purpose it is enough to consider fields satisfying this condition).

An element $y \in F$ is said to be: an *exponential* of a if $y' = a'y$; an *exponential of integral* of a if $y' = ay$; a *logarithm* of a if $y' = a'/a$; an *integral* of a if $y' = a$. In each of these cases, y is defined only up to an additive or a multiplicative complex constant.

Let $K \subset F$ be a differential subfield in F. An element y is said to be an *integral over* K if $y' = a \in K$. An *exponential of integral over* K, a *logarithm over* K, and an *integral over* K are defined similarly.

Suppose that a differential field K and a set M lie in some differential field F.

The *adjunction* of the set M to the differential field K is the minimal differential field $K\langle M\rangle$ containing both the field K and the set M. We will refer to the transition from K to $K\langle M\rangle$ as *adjoining* the set M to the field K.

Definition 1. A differential field F is said to be a *generalized elementary extension* of a differential field K if $K \subset F$ and there exists a chain of differential fields $K = F_0 \subseteq \cdots \subseteq F_n \supset F$ such that $F_{i+1} = F_i\langle y_i\rangle$ for every $i = 0, \ldots, n-1$ where y_i is an exponential, a logarithm, or an algebraic element over F_i.

An element $a \in F$ is a *generalized elementary element* over K, $K \subset F$, if it is contained in a certain generalized elementary elementary extension of the field K. The following lemma is obvious.

Lemma 1. *An extension $K \subset F$ is a generalized elementary extension if and only if there exists a chain of differential fields $K = F_0 \subseteq \cdots \subseteq F_n \supset F$ such that for every $i = 0, \ldots, n-1$, either F_{i+1} is a finite extension of F_i, or F_{i+1} is a pure transcendental extension of F_i obtained by adjoining finitely many exponentials and logarithms over F_i.*

2. Functional Differential Fields and their Extensions

Let K be a subfield in the field F of all meromorphic functions on a connected domain U of the Riemann sphere $\mathbb{C}^1 \cup \infty$ with the fixed coordinate function x on $\mathbb{C}^1$. Suppose that K contains all complex constants and is stable under differentiation (i.e., if $f \in K$ then $f' = df/dx \in K$). Then K provides an example of a *functional differential field.*

Let us now give a general definition.

Definition 2. Let U, x be a pair consisting of a connected Riemann surface U and a non constant meromorphic function x on U. The map $f \to df/dx$ defines the derivation in the field F of all meromorphic functions on U (the ratio of two meromorphic 1-forms is a well-defined meromorphic function). A *functional differential field* is any differential subfield of F (containing all complex constants).

The following construction helps to *extend* functional differential fields. Let K be a differential subfield of the field of meromorphic functions on a connected Riemann surface U equipped with a meromorphic function x. Consider any connected Riemann surface V together with a nonconstant analytic map $\pi\colon V \to U$. Fix the function π^*x on V. The differential field F of all meromorphic functions on V with the differentiation $\varphi' = d\varphi/\pi^*dx$ contains the differential subfield π^*K consisting of functions of the form π^*f, where $f \in K$. The differential field π^*K is isomorphic to the differential field K, and it lies in the differential field F. For a suitable choice of the surface V, an extension of the field π^*K, which is isomorphic to K, can be done within the field F.

Suppose that we need to extend the field K, say, by an integral y of some function $f \in K$. This can be done in the following way. Consider the covering of the Riemann surface U by the Riemann surface V of an indefinite integral y of the form $f dx$ on the surfave U. By the very definition of the Riemann surface V, there exists a natural projection $\pi\colon V \to U$, and the function y is a single-valued meromorphic function on the surface V. The differential field F of meromorphic functions on V with the differentiation $\varphi' = d\varphi/\pi^* dx$ contains the element y as well as the field $\pi^* K$ isomorphic to K. That is why the extension $\pi^* K\langle y\rangle$ is well defined as a subfield of the differential field F. We mean this particular construction of the extension whenever we talk about extensions of functional differential fields. The same construction allows to adjoin a logarithm, an exponential, an integral or an exponential of integral of any function f from a functional differential field K to K. Similarly, for any functions $f_1, \ldots, f_n \in K$, one can adjoin a solution y of an algebraic equation $y^n + f_1 y^{n-1} + \cdots + f_n = 0$ or all the solutions $y_1, \ldots, y_n$ of this equation to K (the adjunction of all the solutions $y_1, \ldots, y_n$ can be implemented on the Riemann surface of the vector-function $\boldsymbol{y} = y_1, \ldots, y_n)$. In the same way, for any functions $f_1, \ldots, f_{n+1} \in K$, one can adjoin the n-dimensional $\mathbb{C}$-affine space of all solutions of the linear differential equation $y^{(n)} + f_1 y^{(n-1)} + \cdots + f_n y + f_{n+1} = 0$ to K. (Recall that a germ of any solution of this linear differential equation admits an analytic continuation along a path on the surface U not passing through the poles of the functions $f_1, \ldots, f_{n+1}$.)

Thus, *all above–mentioned extensions of functional differential fields can be implemented without leaving the class of functional differential fields.* When talking about extensions of functional differential fields, we always mean this particular procedure.

The differential field of all complex constants and the differential field of all rational functions of one variable can be regarded as differential fields of functions defined on the Riemann sphere.

3. Classes of Functions and Operations on Multivalued Functions

An indefinite integral of an elementary function is a function rather than an element of an abstract differential field. In functional spaces, for example, apart from differentiation and algebraic operations, an absolutely non-algebraic operation is defined, namely, the composition. Anyway, functional spaces provide more means for writing “explicit formulas” than abstract differential fields. Besides, we should take into account that functions can be multivalued, can have singularities and so on.

In functional spaces, it is not hard to formalize the problem of unsolvability of equations in explicit form. One can proceed as follows: fix a class of functions and say that an equation is solvable explicitly if its solution belongs to this class. Different classes of functions correspond to different notions of solvability.

3.1. Defining classes of functions by the lists of data. A class of functions can be introduced by specifying a list of *basic functions* and a list of *admissible operations*. Given the two lists, the class of functions is defined as the set of all functions that can be obtained from the basic functions by repeated application of admissible operations. Below, we define the class of *generalized elementary functions* and the class of *generalized elementary functions over a functional differential field K* in exactly this way.

Classes of functions which appear in the problems of integrability in finite terms contain multivalued functions. Thus the basic terminology should be made clear. We work with multivalued functions "globally", which leads to a more general understanding of classes of functions defined by lists of basic functions and of admissible operations. A multivalued function is regarded as a single entity. *Operations on multivalued functions* can be defined. The result of such an operation is a set of multivalued functions; every element of this set is called a function obtained from the given functions by the given operation. A *class of functions* is defined as the set of all (multivalued) functions that can be obtained from the basic functions by repeated application of admissible operations.

3.2. Operations on multivalued functions. Let us define, for example, the sum of two multivalued functions on a connected Riemann surface U.

Definition 3. Take an arbitrary point a in U, any germ f_a of an analytic function f at the point a and any germ g_a of an analytic function g at the same point a. We say that the multivalued function φ on U generated by the germ $\varphi_a = f_a + g_a$ *is representable as the sum of the functions* f and g.

For example, it is easy to see that exactly two functions of one variable are representable in the form $\sqrt{x} + \sqrt{x}$, namely, $f_1 = 2\sqrt{x}$ and $f_2 \equiv 0$. Other operations on multivalued functions are defined in exactly the same way. *For a class of multivalued functions, being stable under addition means that, together with any pair of its functions, this class contains all functions representable as their sum.* The same applies to all other operations on multivalued functions understood in the same sense as above.

In the definition given above, not only the operation of addition plays a key role but also the operation of analytic continuation hidden in the notion of multivalued function. Indeed, consider the following example. Let f_1 be an analytic function defined on an open subset V of the complex line $\mathbb{C}^1$ and admitting no analytic continuation outside of V, and let f_2 be an analytic function on V given by the formula $f_2 = -f_1$. According to our definition, the zero function is representable in the form $f_1 + f_2$ *on the entire complex line*. By the commonly accepted viewpoint, the equality $f_1 + f_2 = 0$ holds inside the region V but not outside.

Working with multivalued functions globally, we do not insist on the existence of a *common region*, were all necessary operations would be performed on single-valued branches of multivalued functions. A first operation can be performed in a first region, then a second operation can be performed in a second, different region on analytic continuations of functions obtained on the first step. In essence, this more general understanding of operations is equivalent to including analytic continuation to the list of admissible operations on the analytic germs.

4. Generalized Elementary Functions

In this section we define the generalized elementary functions of one complex variable and the generalized elementary functions over a functional differential field. We also discuss a relation of these notions with generalized elementary extensions of differential fields. First we'll present needed lists of basic functions and of admissible operations.

List of basic elementary functions

1. All complex constants and an independent variable x.
2. The exponential, the logarithm, and the power x^α, where α is any constant.
3. The trigonometric functions sine, cosine, tangent, cotangent.
4. The inverse trigonometric functions arcsine, arccosine, arctangent, arccotangent.

Lemma 2. *Basic elementary functions can be expressed through the exponentials and the logarithms with the help of complex constants, arithmetic operations and compositions.*

Lemma 2 can be considered as a simple exercise. Its proof can be found in [1].

List of some classical operations

1. The operation of composition takes functions f,g to the function $f \circ g$.
2. The arithmetic operations take functions f, g to the functions $f+g$, $f-g$, fg, and f/g.
3. The operation of differentiation takes function f to the function f'.
4. The operation of solving algebraic equations takes functions $f_1, \dots, f_n$ to the function y such that $y^n + f_1 y^{n-1} + \dots + f_n = 0$ (the function y is not quite uniquely determined by functions $f_1, \dots, f_n$ since an algebraic equation of degree n can have n solutions.

Definition 4. The class of *generalized elementary functions of one variable* is defined by the following data:

List of basic functions: basic elementary functions.

List of admissible operations: Compositions, Arithmetic operations, Differentiation, Operation of solving algebraic equations.

Theorem 3. *A (possibly multivalued) function of one complex variable belongs to the class of generalized elementary functions if and only if it belongs to some generalized elementary extension of the differential field of all rational functions of one variable.*

Theorem 3 follows from Lemma 2 (all needed arguments can be found in [1]).

Let K be a functional differential field consisting of meromorphic functional on a connected Riemann surface U equipped with a meromorphic function x.

Definition 5. Class of *generalized elementary functions over a functional differential field K* is defined by the following data.

List of basic functions: all functions from the field K.

List of admissible operations: Operation of composition with a generalized elementary function ϕ that takes f to $\phi \circ f$, Arithmetic operations, Differentiation, Operation of solving algebraic equations.

Theorem 4. *A (possibly multivalued) function on the Riemann surface U belongs to the class of generalized elementary functions over a functional differential field K if and only if it belongs to some generalize elementary extension of K.*

Theorem 4 follows from Lemma 2 (all needed arguments can be found in [1]).

5. Liouville's Theorem

In 1833 Joseph Liouville proved the following fundamental result.

Liouville's Theorem. *An integral y of a function f from a functional differential field K is a generalized elementary function over K if and only if y is representable in the form*

$$y(x) = \int_{x_0}^{x} f(t)\,dt = r_0(x) + \sum_{i=1}^{m} \lambda_i \ln r_i(x), \tag{1}$$

where $r_0, \ldots, r_m \in K$ and $\lambda_1, \ldots, \lambda_m$ are complex constants.

For large classes of functions algorithms based on Liouville's Theorem make it possible to either evaluate an integral or to prove that the integral cannot be "evaluated in finite terms".

Let us outline an inductive proof of Liouville's Theorem.

Definition 6. A function g is a *generalized elementary function of complexity $\leqslant k$* if there is a chain $K = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_k$ of functional differential fields such that $g \in F_k$ and for any $0 \leqslant i < k$ either F_{i+1} is a finite extension of F_i, or F_{i+1} is a pure transcendental extension of F_i obtained by adjoining finitely many exponentials, and logarithms over F_i.

We will prove the following induction hypothesis $I(m)$: *Liouville's Theorem is true for every integral y of complexity $\leqslant m$ over any functional differential field K.* The statement $I(0)$ is obvious: if $y \in K$, then $y = r_0 \in K$. Now let $y' \in K$ and $y \in F_k$. Since $y' \in F_1$, by induction $y = R_0 + \sum_{i=1}^{q} \lambda_i \ln R_i$, where R_0, $R_1, \ldots, R_q \in F_1$. We need to show that y is representable in the form (1) with $r_0, \ldots, r_m \in F_0 = K$ We have the following two cases to consider:

1. F_1 is a finite extension of $F_0 = K$. The statement of induction hypothesis in that case was proved by Abel and is called Abel's Theorem. We will present its proof in the section 6.

2. F_1 is a pure transcendental extension of $F_0 = K$ obtained by adjoining exponentials and logarithms over K. We will deal with this case in section 7.

6. Algebraic Case

In Section 6.1 we discuss finite extensions of differential fields. In Section 6.2 we present a proof of Abel's Theorem.

6.1. An algebraic extension of a functional differential field. Let

$$P(z) = z^n + a_1 z^{n-1} + \dots + a_n \tag{2}$$

be an irreducible polynomial over K, $P \in K[z]$. Suppose that a functional differential field F contains K and a root z of P.

Lemma 5. *The field $K(z)$ is stable under the differentiation.*

Proof. Since P is irreducible over K, the polynomial $\frac{\partial P}{\partial z}$ has no common roots with P and is different from zero in the field $K[z]/(P)$. Let M be a polynomial satisfying a congruence $M\frac{\partial P}{\partial z} \equiv -\frac{\partial P}{\partial x} \pmod{P}$. Differentiating the identity $P(z) = 0$ in the field F, we obtain that $\frac{\partial P}{\partial z}(z)z' + \frac{\partial P}{\partial x}(z) = 0$, which implies that $z' = M(z)$. Thus the derivative of the element z coincides with the value at z of a polynomial M. Lemma 5 follows from this fact. □

Let $K \subset F$ and $\hat{K} \subset \hat{F}$ be functional differential fields, and P, $\hat{P}$ irreducible polynomials over K, $\hat{K}$ correspondingly. Suppose that F, $\hat{F}$ contain roots z, $\hat{z}$ of P, $\hat{P}$.

Theorem 6. *Assume that there is an isomorphism $\tau\colon K \to \hat{K}$ of differential fields K, $\hat{K}$, which maps coefficients of the polynomial P to the corresponding coefficients of the polynomial $\hat{P}$. Then τ can be extended in a unique way to the differential isomorphism $\rho\colon K(z) \to \hat{K}(\hat{z})$.*

Proof of Theorem 6 could be obtain by the arguments used in the proof of Lemma 5.

6.2. Induction hypothesis for an algebraic extension. Let $z_1, \dots, z_n$ be the roots of the polynomial P given by (2) and let $F_1 = K\langle z_1\rangle$. Assume that there is an element $y_1 \in F_1$ such that $y_1' \in K$, $M_i \in K[x]$ and y_1' is representable in the form

$$y_1' = \sum_{i=1}^{q} \lambda_i \frac{(M_i(z_1))'}{M_i(z_1)} + (M_0(z_1))'. \tag{3}$$

Abel's Theorem. *Under the above assumptions the element y_1' is representable in the form* (1) *with polynomials M_i independent of z_1, i.e., with $M_0, M_1, \dots, M_q \in K$.*

Proof. Let y_1 be equal to $Q(z_1)$, where $Q \in K[z]$. For any $1 \leqslant j \leqslant n$ let y_j be the element $Q(z_j)$. According to Theorem 6 the identity (3) implies the identity

$$y_j' = \sum_{i=1}^{q} \lambda_i \frac{(M_i(z_j))'}{M_j(z_1)} + (M_0(z_j))'. \tag{4}$$

Since $y_1' \in K$ we obtain n equalities $y_1' = \dots = y_n'$. To complete the proof it is enough to take the arithmetic mean of n equalities (4). Indeed the elements $\tilde{M}_i = \prod_{1 \leqslant k \leqslant n} M_i(z_k)$ and $\tilde{M}_0 = \sum_{1 \leqslant k \leqslant n} M_0(z_k)$ are symmetric functions in the roots of the polynomial P thus $\tilde{M}_0, \dots \tilde{M}_q \in K$. □

Remark. The proof uses implicitly the Galois group G of the splitting field of the polynomial P over the field K. The group G permutes the roots $y_1, \ldots, y_n$ of P. The element $\tilde{M}_i = \prod_{1\leqslant k\leqslant n} M_i(z_k)$ and $\tilde{M}_0 = \sum_{1\leqslant k\leqslant n} M_0(z_k)$ are invariant under the action of G thus they belong to the field K.

7. Pure Transcendental Case

In this section we prove induction hypothesis in the pure transcendental case. First we will state the corresponding Theorem 7 and will outline its proof.

Let F_1 be a functional differential field obtained by extension of the functional differential field K by adjoining algebraically independent over K functions

$$y_1 = \ln a_1, \ \ldots, \ y_k = \ln a_k, \ z_1 = \exp b_1, \ \ldots, \ z_n = \exp b_n, \tag{5}$$

where $a_1, \ldots, a_k, b_1, \ldots, b_k$ are some functions from K. We will assume that F_1 consists of meromorphic functions on a connected Riemann surface U and the differentiation in K_1 using a meromorphic function x on U. Let X be the manifold $U \times G$, where $G = \mathbb{C}^k \times (\mathbb{C}^*)^n$. Consider a map $\gamma\colon U \to \mathbb{C}^k \times (\mathbb{C}^*)^n$ given by formula

$$\gamma(p) = y_1(p), \ldots, y_k(p), z_1(p) \ldots, z_n(p),$$

where the functions y_i, z_j are defined by (5).

Let X be the product $U \times (\mathbb{C})^k \times (\mathbb{C}^*)^n$. Denote by $\Gamma \subset X$ the graph of the map γ. Consider a germ Φ of a complex valued function at the point $a \in X$.

Definition 7. We say that Φ is a *logarithmic type germ* if Φ is representable in the form $\Phi_a = R_0 + \sum_{i=1}^q \lambda_i \ln R_i$, where R_i are germs at the point $a \in X$ of rational functions of $(y_1, \ldots, y_k, z_1, \ldots, z_n)$ with coefficients in K and λ_j are complex numbers.

Theorem 7. *Let Φ be a logarithmic type germ at a point $a = (p_0, \gamma(p_0)) \in \Gamma$. Then the germ of the function $\Phi(p, \gamma(p))$ at the point $p_0 \in U$ is a germ of an integral over K if and only if Φ is representable in the following form*

$$\Phi(p, y, z) = \Phi(p, \gamma(p_0)) + \sum_1^k c_i(y_i - y_i(p_0)) + \sum_1^n t_j \ln \frac{z_j}{z_j(p_0)}, \tag{6}$$

where c_i, t_j are complex constants.

Theorem 7 proves induction hypothesis in the pure transcendental case. Indeed $\Phi(p, \gamma(p_0))$ is a germ of a function from the field K and according to (5) the identities $c_i y_i = c_i \ln a_i$, $t_j \ln z_j = t_j b_j$ hold. We split the claim of Theorem 7 into two parts.

First we consider the natural action of the group $G = (\mathbb{C}^k) \times (\mathbb{C}^*)^n$ on $X = U \times G$ and we describe all germs of closed 1-forms locally invariant under this action. Corollary 11 claims that each such 1-form is a differential of a function representable in the form (6).

Second we show that if the germ Φ satisfies the conditions of Therem then the germ $d\Phi$ is locally invariant under the action of the group G (see Theorem 16).

7.1. Locally invariant closed 1-forms. Let G be a connected Lie group acting by diffeomorphisms on a manifold X. Let $\pi\colon G \to \mathrm{Diff}(X)$ be a corresponding homomorphism from G to the group $\mathrm{Diff}(X)$ of diffeomorphisms of X. For a vector ξ from the Lie algebra $\mathcal{G}$ of G the action π associates the vector field V_ξ on X. The germ ω_{x_0} at a point $x_0 \in X$ of a differential form ω on X is *locally invariant under the action* π if for any $\xi \in \mathcal{G}$ the Lie derivative $L_{V_\xi}\omega$ is equal to zero.

Lemma 8. *The germ of the differential $d\varphi_{x_0} = \omega_{x_0}$ of a smooth function φ is locally invariant under the action π if and only if for each $\xi \in \mathcal{G}$ the Lie derivative $L_{V_\xi}\varphi$ is a constant $M(\xi)$ (which depends on ξ).*

Proof. Applying "Cartan's magic formula" $L_{V_\xi}\omega = i_X d\omega + d(i_X\omega)$ we obtain that $L_{V_\xi}\omega = 0$ if and only if $d(L_{V_\xi}\varphi) = 0$, which means that $L_{V_\xi}\varphi$ is constant. □

The following theorem characterizes locally invariant closed 1-forms more explicitly.

Theorem 9. *The germ of the differential $d\varphi_{x_0} = \omega_{x_0}$ of a smooth complex valued function φ is locally invariant under the action π if and only if there exists a local homomorphism ρ of G to the additive group $\mathbb{C}$ of complex numbers such that for any $g \in G$ in a neighborhood of the identity the following relation holds:*

$$\varphi(\pi(g)x_0) = \varphi(x_0) + \rho(g).$$

Proof. For $\xi \in \mathcal{G}$ the Lie derivative $L_{V_\xi}\varphi$ is constant $M(\xi)$ by Lemma 8. Let us show that for $\xi \in [\mathcal{G}]$, where $[\mathcal{G}]$ is the commutator of $\mathcal{G}$ the constant $M(\xi)$ equals to zero. Indeed if $\xi = [\tau, \rho]$ then

$$L_{V_\xi}\varphi = L_{V_\rho}L_{V_\tau}\varphi - L_{V_\tau}L_{V_\rho}\varphi = L_{V_\rho}M(\tau) - L_{V_\tau}M(\rho) = 0.$$

Thus the linear function $M\colon \mathcal{G} \to \mathbb{C}$ mapping ξ to $M(\xi)$ provides a homomorphism of $\mathcal{G}$ to the Lie algebra of the additive group $\mathbb{C}$ of complex numbers. Let ρ be the local homomorphism of G to $\mathbb{C}$ corresponding to the homomorphism M.

Consider a function ϕ on a neighborhood of the identity in G defined by the following formula: $\phi(g) = \varphi(x_0)+\rho(g)$. By definition on a neighborhood of identity the function ϕ has the same differential as the function $\varphi(\pi(g)x)$. Values of these functions at the identity are equal to $\varphi(x_0)$. Thus these functions are equal. □

Assume that $X = U \times G$, where U is a manifold and an action π is given by the formula $\pi(g)(x, g_1) = (x, gg_1)$. Applying Theorem 9 to this action we obtain the following corollary.

Corollary 10. *If germ of differential $d\varphi = \omega$ of a smooth complex valued function φ at a point $(x_0, g_0) \in U \times G$ is locally invariant under the action π then in a neighborhood of the point (x_0, g_0) the following identity holds:*

$$\varphi(x, g) = \varphi(x, g_0) + \rho(gg_0^{-1}), \tag{7}$$

where ρ is a local homomorphism of G to the additive group of complex numbers.

Proof. Follows from Theorem 9 since the element gg_0^{-1} maps the point (x, g_0) to the point (x, g). □

Let G be the group $\mathbb{C}^k \times (\mathbb{C}^*)^n$, where $\mathbb{C}$ and $\mathbb{C}^*$ are additive and mulplicative group of complex numbers. We will consider the group $\mathbb{C}^k \times (\mathbb{C}^*)^n$ with coordinate functions $(y, z) = (y_1, \ldots, y_k, z_1, \ldots, z_n)$ assuming that $z_1 \cdots z_n \neq 0$.

Corollary 11. *If in the assumptions of Corollary* 10 *for* $G = \mathbb{C}^k \times (\mathbb{C}^*)^n$ *in a neighborhood of* $(x_0, y_0, z_0) \in U \times (\mathbb{C}^k \times (\mathbb{C}^*)^n)$ *the following identity holds*

$$\varphi(x, y, z) = \varphi(x, y_0, z_0) + \sum_{1 \leqslant i \leqslant k} \lambda_i (y_i - (y_0)_i) + \sum_{1 \leqslant j \leqslant n} \mu_j \ln \frac{z_j}{(z_0)_j},$$

where $\lambda_1, \ldots, \lambda_k, \mu_1 \ldots, \mu_n$ *are complex constants.*

Proof. Follows from (7) since any local homomorphism ρ from the group $\mathbb{C}^k \times (\mathbb{C}^*)^n$ to the additive group of complex numbers can be given by formula

$$\rho(y_1, \ldots, y_k, z_1, \ldots, z_n) = \sum_{1 \leqslant i \leqslant k} \lambda_1 y_i + \sum_{1 \leqslant j \leqslant n} \mu_j \ln z_j,$$

where λ_i and μ_j are complex constants. □

7.2. Vector field associated to a logarithmic-exponential extension. We use the notations introduced in the section 7. Let G be the group $\mathbb{C}^k \times (\mathbb{C}^*)^n$ and let X be the product $U \times G$ consider the map $\gamma \colon U \to \mathbb{C}^k \times (\mathbb{C}^*)^n$ given by the following formula:

$$y_1 = \ln a_1, \ \ldots, \ y_k = \ln a_k, \ z_1 = \exp b_1, \ \ldots, \ z_n = \exp b_n. \tag{8}$$

The map γ satisfies the following differential relation:

$$d\gamma = da_1/a_1, \ldots, da_k/a_k, z_1 db_1, \ldots, z_n db_n.$$

Definition 8. Let V be a meromorphic vector field on X defined by the following conditions. If V_a is the value of V at the point $a = (p, y_1, \ldots, y_k, z_1, \ldots, z_n) \in X$ then $\langle dx, V_a \rangle = 1$, $\langle dy_i, V_a \rangle = a_i'/a_i(p)$ for $1 \leqslant i \leqslant k$, $\langle dz_j, V_a \rangle = b_j'/b_i(p)$ for $1 \leqslant j \leqslant n$.

Vector field V is regular on $U^0 \times G$, where U^0 is an open subset in U which does not contain zeros and poles of the functions $a_1, \ldots, a_k$, nor poles of functions $b_1, \ldots, b_n$, nor zeros and poles of the 1-form dx. By construction the graph $\Gamma = (p, \gamma(p) \subset X$ of the map γ is an integral curve for differential equation on X defined by the vector field v.

The following lemmas are obvious.

Lemma 12. *The vector field* V *is invariant under the action* π *on* X*. For each element* $g \in G$ *the curve* $g\Gamma \subset X$ *of the graph* Γ *of* γ *is an integral curve for* V*.*

Lemma 13. *The field* $K(y, z)$ *of rational functions in* $y_1, \ldots, y_k, z_1, \ldots, z_n$ *over the field* K *is invariant under the action* π *on* X*. For each vector* $\xi \in \mathcal{G}$ *in the Lie algebra* $\mathcal{G}$ *of* G *the Lie derivative* $L_{V_\xi} R$ *of* $R \in K(y, z)$ *belongs to* $K(y, z)$*.*

7.3. Pure transcendental logarithmic exponential extension. We will assume below that the components (8) of γ are algebraically independent over K.

Liouville's principle. *If a polynomial $P \in K[y_1, \dots, y_k, z_1, \dots, z_n]$ vanishes on the graph $\Gamma \subset X$ of the map γ then P is identically equal to zero.*

Proof. If P is not identically equal to zero then the components of γ are algebraically dependent over the field K. □

Theorem 14. *The extension $K \subset F_1$ is isomorphic to the extension of K by the field of rational functions $K(y, z)$ in $(y_1, \dots, y_k, z_1, \dots, z_n)$ over K considered as the field of functions on X equipped with the differentiation sending $f \in K(y, z)$ to the Lie derivative $L_V f$ with respect to the vector field V introduced in Definition* 8.

Proof. By assumption components (8) of the map γ are algebraically independent over K thus each function from the extension obtained by adjoining to K by these components is representable in the unique way as a rational function from $K(y, z)$. By definition the derivatives of the components (8) are coincide with their Lie derivatives with respect to the vector field V. □

The action π of the group $G = \mathbb{C}^k \times (\mathbb{C}^*)^n$ on X induces the action π^* of G on the space of functions on X containing the field $K(y, z)$. The vector field V is invariant under the action π. Thus π^* acts on $K(y, z) \sim F_1$ by differential automorphisms. Easy to see that a function $f \in K(y, z)$ is fixed under the action π^* if and only if $f \in K$, i.e., the group G is isomorphic to the *differential Galois group* of the extension $K \subset F_1$. We proved the following result

Theorem 15. *The differential Galois group of the extension $K \subset F_1$ is isomorphic to the group G. The Galois group is induced on the differential field $K(y, z)$ with the differentiation given by Lie derivative with respect to the field V by the action of G on the manifold $X = U \times \mathbb{C}^k \times (\mathbb{C}^*)^n$.*

Now we are ready to complete inductive proof of the Liouville's Theorem.

Theorem 16. *Let Φ be a logarithmic type germ at a point $a = (p_0, \gamma(p_0)) \in \Gamma \subset X$. If the germ of the function $\Phi(p, \gamma(p))$ on U at the point $p_0 \in U$ is a germ of an integral f over K then the germ of the differential $d\Phi$ at the point $a \in X$ is locally invariant under the action π on X.*

Proof. By the assumption of Theorem the restriction of the function $(L_V\Phi - f)$ on Γ is equal to zero. Since the function $(L_V\Phi - f)$ belongs to the field $K(y, z)$ the function $(L_V\Phi - f)$ by Liouville's principle is equal to zero identically on X. In particular it is equal to zero on the integral curve $g\Gamma$ the vector field V, where g is an element of the group G. Thus the restrictions of function $L_V\pi(g)^*(\Phi - f)$ to Γ equals to zero. Since the function f is invariant under the action π^* we obtain that the restriction on Γ of $L_V(\Phi - \pi^*(g)\Phi)$ is equal to zero. Differentiating this identity we obtain that for any $\xi \in \mathcal{G}$ the restriction on Γ of $L_V(L_{V_\xi}\Phi$ equals to zero. Thus on Γ the function $L_\xi\Phi$ is constant. Lemma 13 implies that the function L_{V_ξ} belongs to the field $K(x, y)$. Thus the functionL_{V_ξ} is a constant on X by Liouville's principle. Thus the 1-form $d\Phi$ is locally invariant under the action π by Lemma 8. Theorem 16 is proved. □

Thus we complete proof of Theorem 7 and the inductive proof of Liouville's Theorem.

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